COORDINATE SYSTEMS

in

ONE and TWO DIMENSIONS.

by

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Presented to the Faculty of
the Graduate School
of
The University of Kansas
In Partial Fulfillment of the Requirements for
The Degree of Master of Arts.

June , 1914.

COORDINATE SYSTEMS.

INTRODUCTION.

1. Definition. Coordinates may be defined in a number of ways depending upon the point of view of the definer. In elementary analytic geometry the coordinate idea is a very narrow one. In it the coordinates of a point in a plane are defined

a) as two numbers, viz. those associated with the distances of the point from two given lines, measured in given directions, or

b) as two numbers, viz. the distance of the point from a given point, and the angle which the line joining these two points makes with a given line. A broader definition is given by Fischer,[1] when he defines coordinates as numbers which determine single-valuedly the position of a geometric element, and which are, vice versa, determined single valuedly by such position, with the further property that the continuous variation of the coordinates, or of the position of the element causes a corresponding variation in the other. Still more comprehensive definitions are given by

a) Scott,[2] who defines coordinates as quantities[3] that determine the position of a geometric element,

b) Muller,[4] who defines coordinates as numbers by means of which a geometric form in (any) space can be determined with respect to a fixed form of the space in question.

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[1] Fischer. Koordinatensysteme. 11.

[2] Scott. Modern Analytical Geometry. 2.

[4] Muller. Encyklopedia der Math. Wissenschaften III1 ~~4~~ 602.

[3] In this article coordinates will be regarded as numbers.

2. Earliest application and development. The different points of view manifested in these definitions are the results of a growth in the coordinate concept, this growth extending over many centuries. Without doubt, the first suggestion of a coordinate system is found in the work of the Egyptian architects[1] forty centuries ago. It has been shown that it was their custom, after drawing the contemplated figures in miniature, to cross line this work and thus to obtain division of their plan into small square regions. By dividing the surface, upon which the work was to be placed, into similar squares they were able to obtain greater accuracy than could otherwise be obtained in their work, by carrying over to any particular square on the surface an enlarged counterpart of the corresponding square in their plan. In 120 B.C. Hipparchus[2] the astronomer, determined the position of places upon the earth by means of their distances from the equator and the meridian through Rhodes; also clearly a simple rectangular system. Shortly after this, Heron applied this same general method to surveying, in the division of land into distinct parts; a scheme which was also used many years later, i.e. about 1000 A.D., by the Italian surveyors when laying out camps and towns. Tropfke remarks in regard to the work of the Italians, that "The entire placing of the city was a right angled coordinate system, which,

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[1] Tropfke. Geschichte der Elementar Mathematik. II:. 407.

[2] Cantor. Vorlesungen uber Geschichte der Math. I:. 357-8.

of course, now and then suffered change due to the behavior of the ground".

Gerbert[1] about 1000 A.D., may have used a sort of coordinate system in picturing certain physical phenomena, but very little is known of his work. The first, however, to apply the coordinate method to the picturing of physical phenomena, of whom we have definite knowledge, is Nicole Oresme[2] (1361). He assumed a certain line as fixed and drew perpendiculars to this line at different points of it, the length of these perpendiculars being determined by the physical phenomenon under consideration. He then joined the end points of these perpendiculars, thus giving us the first known example of the graph of today; he did not, however, consider curves in any quadrant except the first, and he did not associate curves with algebraic equations.

3. Applications to Analytic Geometry[3] The most important application of coordinates has been the correlation[4] of algebra and geometry. It is true that the solution of geometric problems by means of algebra was performed by the Greeks; Apollonius, (250 B.C.-200 B.C.), in particular, obtained many theorems about conic sections by using as axes a diameter of the conic and a tangent to the conic at one of the intersections of the diameter with the conic. Although many of his theorems may be stated analytically, most

1) Cantor. Vorlesungen uber Geschichte der Mathematik. I: 821.

2) Cantor. " " " " " I: 118-20.

Curtze. Die Math. Schriften des Nicole Oresme.

3) Loria. Verhandlungen des III Inter. Math. Congress (1904) 562-74.

4) Fischer. Koordinatensysteme. 5-6.

writers[1] agree that he was not using coordinates in the sense of modern analytic geometry; although the opposite point of view is held by some.[2] It is also true that the use of geometry in the solution of algebraic problems was common among the Arabs.

The correlation between algebra and geometry was very imperfect, however, before the time of Fermat[3] (1639) and Descartes (1637), and only glimpses of this relation had been obtained. Vieta had previously associated numbers with the points on a line (a very important step) but had gone no further. The first to actually see the close relation between <u>geometric curve</u> and <u>algebraic equation</u> was Fermat. The credit is usually given to Descartes, due to the fact that his work was the first published article[4] which emphasised this relation, but Fermat's treatment was both simpler and much more comprehensive. There is some doubt as to the breadth of the coordinate concept in the mind of Descartes, and as to his grasp of the relation between geometric curve and algebraic equation. Thus, e.g. in some of his algebraic work he seems to discard negative roots entirely, whereas in some of his figures he shows curves in more than one quadrant, thus clearly making use of a negative coordinate. Again, nowhere does he give the equation of a straight line, the simplest algebraic equation, whereas Fermat gives it in the form

$$D(R-A)=BE,$$

E and A being variables. The question as to whether Descartes

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1) Tropfke.	Geschichte der Elementar Math.	II:408-9
Cajori.	A History of Mathematics.	48
2) Heath.	Apollonius of Perga	
3) Cantor.	Vorlesungen uber Geschichte der Math.	I:321-2
Cantor.	" " " " "	III:740-4
4) Compare	Oeuvres de Descartes	VI:369ff.

always considered the axes as being at right angles to each other cannot be answered definitely. Although many authors[1] agree that he thought of the angle between the coordinate axes as having any value whatsoever, others disagree[2]. The writer is inclined to the view that in actual work Descartes used rectangular axes only, since in setting up a coordinate system in space he definitely states that the axes are to be perpendicular to each other[3].

The terms "coordinates" and "axis of coordinates" were not used by Descartes, but are due to Leibnitz[4] to whom also we owe the introduction of the terms "abscissa" and "ordinate". Cajori, however, gives the credit to Stefano degli Angeli (1659) for the first use of the term "abscissa", and he also states that Descartes used the term "ordinate".

The work of Descartes was so involved in its nature, and his statements were so peculiar that they did not assist much in the immediate development of the coordinate idea and of the methods of analytic geometry. A real impetus to further development was given in 1692 by Leibnitz[5] when he stated that the coordinates of a point might be considered as determined by the intersection of any two definite curves. Later the coordinates thus obtained were called curvilinear coordinates. In the year previous, Jacob Bernoulli[6] had made the first use of polar coordinates in construct-

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1) Tropfke. Geschichte der Elementar Math. II:16.
Cajori. A History in Mathematics. 185.
Bailey and Woods. Analytic Geometry. 4.

2) Ball. A Short History of Mathematics. 272.
Cantor. Vorlesungen uber Geschichte der Math. II:740.
For an example of the resulting inaccuracy see
Smith. An Elementary Treatise on Anal. Geometry, Preface.

3) Oeuvres de Descartes VI: 440.

4) Müller. Encyklopädie der Math. Wissenschaften. III 4:670.
Cajori. A History of Mathematics. 1.

ing the parabolic spiral;but he did not make any further use of this system. Müller,evidently for this reason,gives the credit for the first general use of these coordinates to Euler(1748),and de Varignon[6] gives such credit to Fermat. The next century saw the extension of these two systems to problems of three dimensions,but no absolutely new systems were developed in that time.

5) Cantor Vorlesungen uber Geschichte der Math. III:204;461-2
6) Müller Encyklopädia der Math.Wissenschaften. III,4: 656-7.

4. Modern Extensions. It was not until the period between 1827 and 1837 that the next real extensions of the coordinate idea were made, and then independently, by Mobius[1], Plucker[2], Chasles[3] and others. These extensions consisted in relating a point in a plane to three fixed lines rather than to two. The new systems thus obtained also had the additional property of making it possible to express any algebraic curve by means of a homogeneous equation in the variables of such a system,and were therefore often called homogeneous coordinates. We may include these systems under the head "trilateral" or "Dreiseits" coordinates. They are the first step in the separation of the coordinate concept from the idea of distance. Von Staudt[4], in 1857,extended these coordinates and made them absolutely independent of distance or measurement of any kind;these coordinates being included under the general name of projective coordinates. It was,as is evident,an easy step to the extension of the above trilateral coordinates to three dimensions under the name

1) Möbius Gesammelte Werke. I:50-60.
2) Plücker Gesammelte Wissen.Abhandlungen. I:124-58.
3) Chasles Geometrie Superieure. 315-26.
" Apercu Historique des Methodes en Geometrie.633-9.
4) von Staudt *Beitrag zu Geometrie de lage(1857).
*References marked with a star are not in the K.U.libraries.

"quadriplanar" coordinates.

Lame[1] in 1833, began the publication of a series of articles which gave a great impetus to the study of curvilinear coordinates. He is usually regarded as the originator of such coordinate systems, in spite of the fact that Leibnitz, Lagrange[2] and Gauss[3] had used them to a more or less degree. Much has been written with regard to curvilinear coordinates and many special types of such coordinate sustems have been developed. Curvilinear and projective coordinates have the common property that both are independent of the idea of distance, a remarkable coincidence considering the fact that curvilinear coordinates result from the study of the theory of functions, and projective coordinates from the study of pure geometry.

An entirely new line of work, productive of many results, readily followed from the principle of duality, developed in other lines of geometry by Gergonne[4] (1825-6) and Poncelet. The activity in this new line of work was accelerated by Plücker[5] when he considered the line instead of the point as the fundamental element in the plane. Systems in the plane, dual to many of those previously developed now began to be worked out, among them the duals of the Cartesian[6] and the polar[7]. In a similar way, the space duals of nearly

1) Todhunter. The Functions of Laplace, Lame, etc. 210.
2) ——— Oeuvres de Lagrange. III:580ff; 624.
3) Gauss. Werke. VI:408-43.
4) Merriman and Woodward. Higher Mathematics. 561.
5) Plücker. Gesammelte Wissen. Abhandlingen. I:178-9.
6) Schlegel. *Association Fr. pour l'avance. des Sciences (1885): 156-.
Schwering. *Zeitschrift fur Math. und Physik. (1876) XXI: 278-86.

all the former coordinate systems were obtained. Coordinate systems were also extended to n-dimensions[1] and there dualized. Not only the point, but the line, plane, circle[2], or any other geometric form were now in turn taken as the element, and the coordinates found with respect to some fixed geometric form, in any dimension whatever; e.g. the position of an r-space (the element) in an n-space may be determined by its position with respect to a fixed geometric form in the n-space. The number of coordinates necessary to determine the given r-space will equal the dimension of the n-space when the r-space is taken as the element.

As early as 1852[3] the position of a point in a plane was determined by means of its distances from two points, and later, from three or more, or by powers of those distances[4]. In the same way the position of a point in a plane was determined by means of the two[5] or more angles made with given fixed lines by the lines joining the points in question to given fixed points.

Finally, from the investigations of the foundations of geometry, especially of projective geometry, coordinate systems which will hold for any finite geometry[6] have been set up; coordinate systems now being established[7], which are independent of the usual assumptions of order and continuity.

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1) Grassman. Ausdehnungslehre.

2) Thomae Grundriss einer anal. Geometrie der Ebene. 47.

3) Salmon. Higher Plane Curves (1852); in new ed. page 123-4.

4) Casey. A Treatise on the Anal. Geometry of the etc. 301,61.

5) Mitchell. Geom. and Collinear Groups of the Finite Proj. Plane

6) Veblen and Young. Projective Geometry. I:26-9

CLASSIFICATION.

In the following, the basis for classification is the dimension of the space with regard to the geometric element taken as the fundamental element of the space in question. This will vary for a given space with different elements, e.g. if the point is taken as the element, then ordinary space is three dimensional, but if the line is taken as the element, then ordinary space is four dimensional. The principle of duality will be assumed to hold for all those systems for which a dual exists; however the proof given by Veblen and Young[1] will apply directly to many of the systems given. By this ~~system~~ principle a coordinate system which has been set up for a given geometric form of any dimension may immediately be dualized so as to apply to any geometric form of the same dimension. But since this principle does not apply to purely metric properties, there are certain systems whose duals do not exist.

I: ONE DIMENSIONAL PRIMITIVE FORMS.

The one dimensional primitive geometric forms are[2]

(a): the points on a line

(b): the planes on a line

(c): the lines on a point and on a plane, or a flat pencil of lines.

According as the point, the line, or the plane is the element we have the point, line or plane coordinates.

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1)	Veblen and Young.	Projective Geometry.	I: 15–33.
	Sturm.	Die lehre v.d. geometrischen Verwandt.	II: 9–10.
	Scott.	Modern Analytic Geometry.	51–5.
2)	Fiedler.	Die Darstellende Geometrie.	I: 108–10.

1. The Cartesian system. The simplest coordinate system of points on a line is the Cartesian[1], also known as the abscissa, or "gewohnlich" (customary"[2]) coordinate system, which was invented by Vieta.[2)] In this system the distance of any point from a given point, the origin, measured in a given directiom and with a certain unit of length, determines the coordinate of the point. Positive numbers are associated with points on one side of the given point, and negative numbers with those on the other side. This one-to-one correspondence [3] between the points on a line and the numbers of the coitinuum between $-\infty$ and $+\infty$ is of fundamental importance[4] in setting up this coordinate system; and in the majority of coordinate systems a similar one-to-one correspondence between the sets of coordinates and the geometric elements exists. But this is not always true, e.g., in the elliptic more than one element may be associated with a single set of numbers, and in the homogeneous system more than one set of numbers may be associated with the same element.

The corresponding system to which the name "gewohnlich"[5] is also given, for the lines in a flat pencil considers the angle which any given line makes with a fixed line of the pencil as the coordinate of the line. A similar system evidently can be set up for a pencil of planes, the number associated with the measure of the dihedral angle between any plane and a fixed plane of the pencil being the

1) Heffter u. Koehler. Lehrbuch der Analytischen Geometrie. I:29–31.

2) Pascal. Repertorium der Höheren Mathematik. II_2 9,45,86.

3) By one-to-one correspondence we shall mean a one to one correspondence that is one-to-one both ways. A correspondence which is one-to-one but one way will be so designated.

4) Fischer. Koordinatensysteme. 6–16.

5) Pascal. Repertorium der Höheren Mathematik. II: 16.

coordinate of the plane. As is evident, there is no one-to-one correspondence in these systems. A line coordinate system which does have this correspondence is the tangent system[1], in which the coordinate of any line of a pencil is given by the tangent of the angle which it makes with a fixed line of the pencil. Moreover, by cutting a pencil of lines by a line perpendicular to the fixed given line, we obtain an abscissa system on the line by giving to each point the coordinate of that line of the flat pencil which goes through it. A generalization of this system will be given later.

2. Parameter and ratio systems. A slightly different system is obtained when we take as the coordinates of a point, its distance from the origin, multiplied by a fixed constant, or parameter. This enables us to make any point the unit point, no matter where the origin is and what the unit of length. This system is known as the parameter[2] system and is equivalent to simply changing the unit of length.

Again, the coordinate of a point on a line may be taken as the ratio of its distances from two given points on the line[3] and also as the ratio of the two distances, each multiplied by a definite constant[2]; the latter being then a parameter system. The analytic relations connecting this more general parameter system and the abscissa system are

$$x' = m\,\frac{x-f}{x-g}\ ,\text{or}\qquad x=\frac{mf-gx'}{m-x'}\ ,$$

where x' and x are the coordinates of the point in this parameter

[1] Plücker. *Systeme der Analytischen Geometrie (1835). 104.
Heffter u. Koehler. Lehrbuch der Analytischen Geometrie. I:101-6.
Castlenuovo. Lezione di geom. anal. e prozettiva. (Rome, 1905)

[2] Sturm. Die Lehre v.d. Geom. Verwandschaften. I:3-23.

[3] Heffter u. Koehler. Lehrbuch der Analytischen Geometrie. I:34-5

system and in the abscissa system respectively, f and g being the coordinates of the two points from which the distances are measured, and m is the ratio of the two constants by which the distances were multiplied. The corresponding systems for the pencil of lines, or planes, are set up in a similar way, but instead of the ratio of two distances, the coordinate is given in each case by the ratio of the sines[1] of the angle formed by any line, or plane, with two fixed lines, or planes, of the pencil, or by the ratio of those sines multiplied by constants.

3. Cross ratio or Projective Systems. The cross ratio of four fixed points on a line is a definite number[1]. If, however, we consider only three of the points as fixed and the fourth as variable, it is very evident that there will be a one-to-one correspondence between the points of the line and the values of the cross ratios of such points with respect to the three fixed points on the line. This enables us to consider the values of their cross ratios as the coordinates of the various points on the line. Thus, if A, E, and P are any three distinct collinear points, the cross ratio coordinate x of the point X on the line AEP is

$$x=(AEPX)=\frac{AP}{PE}:\frac{AX}{XE}=\frac{AP}{EP}\cdot\frac{EX}{AX}.$$

It is evident at once from the fact that $\frac{AP}{EP}$ = a constant, that this system is a type of the general parameter system above. Moreover, if the points A, E, and P are taken as the infinitely distant point, the unit point, and the origin, then

$$(\infty 0 1 X)=\frac{\infty-1}{1-0}\cdot\frac{X-0}{\infty-X}=x\;;$$

in other words, if we project from a point Q the points of the line AEP upon a line parallel to a AQ and take as the origin the projection of E, and as the unit point the projection of P, then the coordinate of the projection of X will be its distance from the origin.

1) Scott. Modern Analytical Geometry. 38,144ff.

Thus we have established the projective relation betwee the cross ratio system and the abscissa system[1] Although there is this apparent dependence of the cross ratio[2], "Wurf" or throw[3], or double ratio[4] coordinate system upon distance, such systems have been set up entirely independent of the idea of distance[5]. The cross ratio of four points being an invariant under projection[6], this system is clearly a very useful one in projective geometry; these coordinates for this reason being often known as projective[7] coordinates.

A special system of projective coordinates arises when one of the three fixed points bisects the segment joining the other two. To such a system the name "barycentric" has been applied[8].

1) Van der Vries.
2) Fischer. Koordinatensysteme. 36-48.
3) Vahlen. Abstrakte Geometrie. 124ff.
4) Heffter u. Koehler. Lehrbuch der Anal. Geom. I:44-5.
5) v. Staudt. *Beitrage zu Geom. d. Lage (1857).
Veblen and Young. Projective Geometry. I:...
6) Scott. Modern Analytical Geometry. 144-8.
7) Pascal. Repertorium der Hoheren Mathematik. II:127-34.
8) Pascal. " " " " II$_2$:10.

Projective coordinate systems may also be established for pencils of lines or planes, either from the standpoint of distance, or that of projection[1]. We will give the two methods for the pencil of lines; these being easily extended to a pencil of planes.

(a). Let O be the vertex of a pencil of lines and let a, e and p be three fixed distinct elements; then the cross ratio coordinate of any element x of the pencil is given by

$$(aepx) = \frac{\sin(ap)}{\sin(pe)} : \frac{\sin(ax)}{\sin(xe)}$$

where (ap) denotes the angle between the lines *a* and *p*, as in Fig.1.

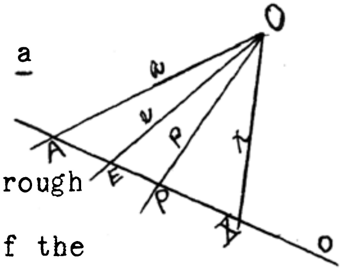

Fig.1.

(b) Cut the pencil by a line *o*, not passing through the vertex *O*, and consider any arbitrary line of the pencil as associated with that point of *o* in which the line *l* cuts *o*. Fix any coordinate system whatever on the line *o*; this associates with every point of the line *o* a definite coordinate and by the association just made this gives to every line of the pencil a definite coordinate. Thus, if the lines *a*, *e*, *p* and *x* of the pencil cut the line *o* in points having in any system whatsoever the coordinates *A*, *E*, *P* and *X* respectively, it is evident that the coordinate of the variable line *x* may be taken as equal to (AEPX). The two methods, however, are in reality the same[1]; and in this way we may obtain a projective line, or plane, system from any point system whatsoever, and vice versa.

An interesting type of projective coordinates is the following application of the tangent coordinates to points on a line. Associate with each line of a pencil on a point *P* the tangent of the angle it makes with the fixed line OP. Let the coordinate of any point *X'* on a coplanar line OX' (not through P) be given by the coordinate of the line PX', as in Fig.2. Thus a projective system has been set up on the line OX'. If OX be perpendicular to OP at O an abscissa system, as has already been stated is the result. The relation between the abscissa coordinate *x* and the projective coordinate *x'* is determined as follows;-

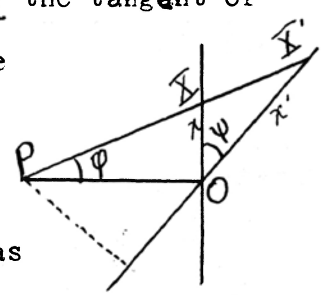

Fig.2.

[1] cf. Scott. Modern Analytical Geometry. 154-6

$$OX'P = 180^\circ - (\varphi + \psi + 90^\circ) = (90^\circ - \psi) - \varphi$$

$$\frac{\sin\varphi}{\sin OX'P} = \frac{OX'}{OP}$$

then, $$x' = OX' = \frac{OP\sin\varphi}{\sin(90^\circ - \psi) - \varphi}\,.$$

$$= \frac{OP\ \sin\varphi}{\sin(90^\circ - \psi)\cos\varphi - \cos(90^\circ - \psi)\sin\varphi}\,.$$

$$= \frac{x}{\cos\psi - \tan\varphi\sin\psi}\,.$$

If we, for simplicity, take $OP = 1$ we obtain

$$x' = \frac{x}{\cos\psi - x\sin\psi}\,.$$

It may be shown that this bilinear expression which gives the analytical relation between these two projective systems(for the abscissa system is a special tpye of the projective system) is not a mere coincidence; and that the relation which holds between any two kinds of projective coordinates will be given by some type of bilinear equation.

It may be shown that in any system so far established any element of any one of the three one dimensional forms may be represented by a non-homogeneous equation in one variable. Any equation of degree n in this variable, then, will represent the n elements of the form under consideration. For this reason these coordinate systems have been known as <u>non-homogeneous</u> systems.

4. Homogeneous systems. Other systems, known as homogeneous, may, however, be obtained from these non-homogeneous systems. Thus, if x be the coordinate in a one dimensional form we may write $X = x_1 : x_2$ and call x_1 and x_2 the homogeneous coordinates of the ~~point~~ element[1]. It should, however, be noted that in homogeneous systems the ratios only of the coordinates are considered, e.g. (kx_1, kx_2) and (x_1, x_2) are the homogeneous coordinates of the same ~~point~~ element. By the above substitution a non-homogeneous equation in x is made homogeneous in x_1 and x_2, and it is for this reason that such coordinates are called homogeneous. Evidently to every non-homogeneous system there is a corresponding homogeneous system. The homogeneous system corresponding to the Cartesian is known as the Hessian system[2].

Geometric significance may also be given to homogeneous coordinate systems; thus, in the ratio system, the homogeneous coordinates of the point are proportional to its distances from the two fixed points; in the double ratio system, where the coordinate of X was given by

$$(AEPX) = \frac{AP}{PE} : \frac{AX}{XE} = \frac{AP.EX}{EP.AX},$$

the homogeneous coordinates are proportional to the distances of the point X from the two fixed points E and A multiplied by the constants AP and EP respectively.

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1) Möbius. Gesammelte Werke. I:51.

Pascal. Repertorium der Hoheren Mathematik. II$_1$ 10:111,17-9.

2) Veblen and Young. Projective Geometry.

5. Curvilinear coordinates. The coordinate systems in one dimension may be extended in some cases to include the points on a curved line, and dually to sets of lines (or planes) not on the same point (or line). We will include such systems under the term curvilinear coordinates, the name being taken from their analogy to curvilinear coordinates in two dimensions.

A large number of these systems are projective in character and will be obtained by establishing one-to-one correspondences.[1] For example, if we take a pencil of lines with a vertex on a conic,[1] we may give to each point on the conic the coordinate associated with that line of the pencil which passes through it; likewise for a line conic, or a cone of lines or planes. Coordinates may be given to the tangents of a line conic in the following way.[2] Consider m as a parameter in the equation

$$am^2 + bm + c = 0$$

where a, b and c are linear functions of the Cartesian variables x and y. For any given value of m the above equation represents a straight line; all the lines thus obtainable, however, being tangent to the conic section

$$b^2 - 4ac = 0$$

The value of m may, therefore, be taken as the coordinate in the * of lines tangent to the conic. This method may be extended to higher[3]

[1] Veblen and Young. Projective Geometry.

[2] cf. Darboux. Jahr. uber die Fort. der Math. IV (1872) 319.
Salmon. Conic Sections. 246ff.

[3] Bassett. Elementary Treatise on Cubic and etc. 8-12.

plane curves, e.g. if a curve of the n th degree has an (n-1)-tuple point, then any line through this point will cut the curve in one, and only one, other point.[1] If we establish a coordinate system on the pencil of lines with the vertex at the (n-1)-tuple point, and give to each point on the curve the coordinate of the line passing through it, we will have set up a coordinate system for the points on the curve, with the possible exception of the (n-1)-tuple point. A more general method may be given, enabling us to establish such a system on any curve having its maximum number of double points[2] i.e. a unicursal curve. We will give the method for the quartic having three double points. Describe a conic through the three points, A, B, C in Fig. 3, and through one other fixed point D on the quartic. Any such conic will meet the quartic in one, and only one, other point. To every point on the quartic there will be conversely, one and only one such conic passing through it. If at one of the four fixed points, tangents are drawn to this pencil of conics, there will be a one-to-one correspondence between the points on the quartic and the pencil of lines; i.e. the coordinate of a point X will be the coordinate of the tangent (at D say) to the conic passing through ABCDX. Evidently to every kind of coordinate system for lines on a point there will be a corresponding system for points on a unicursal curve. This method may also be dualized to include class curves having their

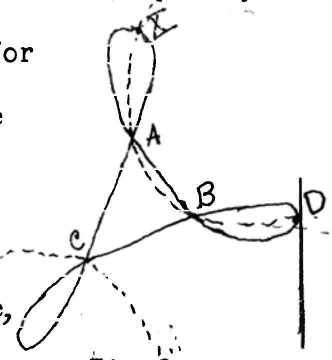

Fig. 3.

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1) Salmon. Higher Plane Curves. 28-9.

2) Clebsch. Vorlesungen uber Geometrie.

maximum double tangents and finally to various kinds of cones.

Another system, which by analogy to its use in two dimensions may be called the natural system[1], may also be established for a continuous unipartite curve with one and only one infinitely distant point. The coordinate of a point in this system will be given by its distance from a fixed point, known as the origin, the distance being measured along a curve, positive values being given to points on one side of the origin, and negative values to those on the other. The analogy between this system and the abscissa system is apparent.[2] Systems analogous to the ratio, parameter and cross ratio systems may also be established for the points on such curves and this method also dualized.

~~6. Curvilinear coordinates The coordinate system in one~~

6. Generalization. It has been shown that the relation between any two projective coordinates systems, which includes all those systems thus far considered, which have to do with primitive one dimensional forms, must be of the form

$$x' = \frac{ax + b}{cx + d}$$

where a, b, c and d are constants, and x and x' are the coordinates in the two systems. In other words, if x is the coordinate of any element in a projective system, this equation gives the coordinate of the same element in some other projective system whose fundamental elements depend upon the particular values of a, b,

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1) Weinleiter. Spezielle ebene Kurven. 169-80.
Müller. Encyklopädie der Math. Wissen. III_1 4:634.
2) Böger. Ebene Geometrie der Lage. 79ff.

c and d in the case under consideration.[1] We may, however, consider a more general system for a primitive one dimensional form in which we define x', the new coordinate, as equal to the quotient of the two functions $f(x)$ and $F(x)$ where x is the coordinate of the element in some system already defined, and $f(x)$ and $F(x)$ are continuous algebraic functions of this coordinate, it being understood that they have no common factor. If $f(x)$ and $F(x)$ are linear, this clearly reduces to the projective system above. In general, the one-to-one correspondence hitherto noticed between the numbers of the contimuum and the elements will be lacking in these generalized one dimensional coordinates. It is possible, however, in some cases where this one-to-one correspondence does not exist, to establish it by imposing extra conditions; this can be illustrated by the following example. Let

$$\text{Let} \qquad f(x) = x^3 \quad , \qquad F(x) = 1$$

then $\qquad x' = x^3 \qquad$ and it is evident that to every value of x there will be one and only one value for x', but on the other hand to every x' there will be three corresponding values of x. If, however, we make the additional assumption that only real values of x are to be considered, it is evident that the correspondence will be one-to-one. Generalized systems of the kind mentioned in this paragraph may also be extended to the consideration of such other one dimensional forms, not primitive, as were discussed in the previous paragraph.

Thus far, only points, lines and planes have been considered as

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[1] Veblen and Young Projective Geometry. I:

elements. Any curve or surface may, however, be taken as the element in a similar way. Thus, if the circle with center at the origin is considered as an element, we may obtain a one dimensional family of circles and the position of any circle will be determined by one non-homogeneous coordinate, viz., the radius of the circle. Likewise for the pencil of conics passing[1] through four fixed non collinear points, say of the form $k_1 + \lambda k_2 = 0$, one and only one conic of the pencil is determined by each value of λ and vice versa. The coordinates of any conic of the pencil may be assumed to be the value of λ which is associated with it, or if we wish to generalize, it may be taken to be some definite function of λ. Obviously higher plane curves, surfaces, etc. may be used in the same way[2].

Other geometric one dimensional forms are found in a space of *n* dimensions and the previous coordinate systems may be extended so as to include these[3]. One dimensional coordinate systems are also used in non-euclidean geometry[4], and in particular in finite geometry[5].

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1)	Salmon.	Conic Sections.	(1869).	299.
2)	Muller.	Encyklopadie der Math. Wissen.	III[1]	4:721.
3)	Bertini.	Introduzione alle Geometria.		
4)	Coolidge.	Non euclidean Geometry.		64.
	Cox.	Quarterly Jour. of Math.	XVIII(1882)	178ff.
5)	Dickson.	Linear Groups.		54-88.

II. TWO DIMENSIONAL PRIMITIVE FORMS.

We will conclude this article by a consideration of the coordinate systems connected with the primitive two dimensional forms[1], viz.,

(a): the points on a plane

(b): the planes on a point, or bundle of planes

(c): the lines on a point, or bundle of lines

(d): the lines on a plane;

the coordinate systems connected with (a) above will be considered in detail, and with each system will be associated as many extensions to the other primitive two dimensional forms as have been made.

A. LINEAR.

Linear[2] coordinates in two dimensions will be considered as including all those systems in which the equation of a line is given by a linear equation, and also the duals of such systems. This classification is a modification of that ~~done~~ given by Müller.[3]

1. Cartesian coordinates. The Cartesian coordinates[4] of a point are the distances of the point from two intersecting lines, the distance from each line being measured parallel to the other. The lines are called the axes[5], or axes of coordinates; their inter-

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1) Veblen and Young. Projective Geometry. I:.
 Fiedler. Die Darstellende Geometrie. III:72-5.
2) Bonnycastle. Inductive Geometry.* (1834): 203-6.
3) Muller. Encyklopadie der Math. Wissen. III[1]4:634-5.
4) Casey. A Treatise on the Analytic Geom. etc. 4ff.
5) Ashton. Analytic Geometry. 7-9.

* Charlottesville, Va.

section, the origin. The distance from one line is called the abscissa, or x coordinate, and the distance from the other line the ordinate, or y coordinate. The two numbers associated with these distances are known as the coordinates of the point. These coordinates are known among German writers as parallel coordinates[1] other terms which have been used synonymously for these are:-

(a)	rectilineal	Coffin	Elements of C.S. and Anal.Geom.	33.
(b)	rectilinear	Hutton	A Course in Math. (1843).	247.
(c)	"geradelinige"	Smith	Coordinate Geometry.	13-4.
(c)	bilinear	Newcomb	Analytic Geometry.	13.
(d)	projective	Booth	*Some New Geom. Methods.	passim.
(e)	general	Johnson	Analytic Geometry.	21.
(f)	"gerade Linien"	*Klügel	*Math. Wörterbuch.	556.
(g)	"gemeine"	Staude	*Analytische Geometrie.	49.

all of them being often referred to simply as ordinary, "gewöhnliche" or usual coordinates. If the angle between the axes is 90°, the coordinates are known as rectangular, or orthogonal coordinates; otherwise, as oblique, or oblique angled[3] coordinates.

The units of measurement on the two axes need not be the same. If they are the same, the coordinates are known as isoscles Cartesian (or parallel) coordinates[4] if not, as general Cartesian

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1) Mobius. Gesammelte Werke. (1827). I:79.

2) Pascal. Repertorium der Höheren Math. II,2:37.

3) Pilgrim. *Math.natur. - - - - Mittheil. (Ser.2.) IX(1907): 53-64. 63-87.

4) Heffter u.Koehler. Lehr.der Analytischen Geom. I:244 note

(Leipzig + Berlin, 1905)

coordinates. A Cartesian system is necessarily non-homogeneous. It can be made a homogeneous system by substituting $x_1 : x_3$ for x and $x_2 : x_3$ for y. If this is done, the numbers x_1, x_2, x_3 are called homogeneous Cartesian coordinates. This system has also been named the Hessian [1] system, after its inventor, Hesse.

Rectangular coordinates have been used for many years to determine the position of a point in the complex plane. They are also applicable to problems on the surface of the earth, as long as the area considered is small enough to make the amount of curvature small. Other applications of these coordinates to scientific investigations are too numerous to mention.

A system of coordinates of a line, known as vector coordinates arises from the Cartesian coordinates in a simple manner.[2] If $(x_1, y_1), (x_2, y_2)$ are the Cartesian coordinates of any two given po points, we may call the values x', y' where

$$\begin{cases} x' = x_2 - x_1 \\ y' = y_2 - y_1 \end{cases}$$

the coordinates of the directed segment having (x_1, y_2) and (x_2, y_2) as initial and terminal points.

The general application of Cartesian coordinates to elementary geometry was not accomplished at an early date, for it was not until the time of Euler that question of signs was clearly understood.[3]

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1) Heffter u. Koehler. Lehr. der anal. Geometrie. I:201-3.
cf. Müller. Encyklopädie der Math. Wissen. III$_1^4$:608.
Frankenback. *Uber ein neue Koor.-Systeme. Rostock(1869).

2) Müller. Ency. der Math. Wissen. III$_1^4$:605-9.

3) " " " " " " 606_n.

Loria. Verhand. des Inter. Math. Cong. (1904): 567-8.

Moreover, the coordinate process was at first used only as an auxiliary, and secondary, method for developing the properties of conic sections;[1] preeminence being given them for the first time about 1850[2]. The general use of coordinates in the elementary theory of equations has been of very recent date[3].

2. <u>Systems dual to the Cartesian.</u>

(a) <u>Plücker line</u> coordinates. In a number of systems which have considered the duals of the Cartesian, the duality is not always complete. Mobius[4] (1827) had a general idea of a line system, but Plücker (1829) was the first to obtain the relation

$$1 + a\varphi + b\psi = 0$$

between point and line coordinates. He considered as the coordinates of a line the negative ~~negative~~ reciprocals of its intercepts upon the two axes. This system is generally used by the Germans and is known as the <u>Plucker</u> coordinate system.[6] It is often combined with the Cartesian to form the Cartesian-Plucker system,

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1) Hutton. A Course in Mathematics (1813) and (1843).

2) Davis. Elements of Analytic Geometry (1847) and (1853).

3) Beman and Smith. Academic Algebra. (1902).

Young and Jackson. Elementary Algebra. (1908). 210-3.

4) Möbius. Gesammelte Werke. I:51-2.

5) Plücker. Gesammelte wissen. Abhandlungen. I:[illegible]-9.

6) Fiedler. Die Darstellende Geometrie. III:91.

which is a point and line coordinate system. These coordinates are also known as bilinear tangential coordinates.[1]

(b) Boothian tangential coordinates. The principle of duality makes a line correspond to a point and accordingly enables us to consider any curve in the plane as the envelope of its tangents instead of as the locus of its points. For this reason English writers have been lead to call line coordinates "tangential coordinates". In this article, however, the term "line coordinates" will be used. Booth's[2] system, given in 1840, was obtained independently of Plücker's previous development. In fact, Booth established his system from geometric considerations, whereas Plücker began with algebraic considerations, and later developed the geometric relations. Booth states his system as follows:-

"The general equation of a straight line in a plane, in Cartesian coordinates is

$$\frac{x}{a} + \frac{y}{b} = 1$$

In this equation a and b denote the intercepts on tha axes cut off by the straight line, while x and y are the variable current coordinates of any point moving along this line. Now if we fix this point making x and y constant and suppose a and b to vary instead, we shall have the means of defining this point". He then puts

$$a = \frac{1}{\xi}, \quad b = \frac{1}{\eta}$$

and obtains the symmetric form

$$a\xi + b\eta = 1$$

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1) Loud. * The Analyst. X (1883): 50-3.

2) Booth. * Some New Geomertic Methods. I: 1ff.

which is the tangential equation of the point in the variables $(\xi\eta)$. As is evident the only difference between this system and Plückers system is that the reciprocals, instead of the negative reciprocals, of the intercepts on the axes are taken as the coordinates of the line. Englishmen in general, however, have until recently used this system rather than that of Plücker, and have called it the Boothian tangential system.[1]

Of the two systems here considered, the Plücker is the one used at the present time, due to its greater symmetry when used with its allied homogeneous system.

(c) Ferrers.[2] "There is another system of tangential coordinates which bears a close analogy to the ordinary Cartesian system. If x, y be the Cartesian coordinates of a point, referred to two rectangular axes, the the intercepts on these axes of the polar of the point with respect to a circle whose center is the origin and radius k will be $\frac{k^2}{x}, \frac{k^2}{y}$ respectively. These intercepts completely determine the position of the line, and their reciprocals may be taken as its coordinates, and denoted by the letters ξ, η." Ferrers calls these the tangential rectangular coordinates of the line. If we take k = 1 then

$$\begin{cases} \xi = x \\ \eta = y \end{cases};$$

and the ~~rectangu~~ coordinates of a point in rectangular coordinates will be the same as the tangential rectangular coordinates of its

1) Bassett. Elem. Treatise on Cubic and Quartic Curves, 30-1.
Ferrers. Trilinear Coordinates. 141-2.

2) " " "

polar with respect to a unit circle. These two systems form a simultaneous point and line system, the relation between them being

$$\xi x + \eta y = 1 .$$

For a long time these systems were regarded as the duals of the Cartesian.[1] It was seen, finally, that this duality was not exact[2] there being no definite point in the plane which corresponds in this system to the infinitely distant point in the Cartesian system. Moreover, in a system dual to the Cartesian it would be necessary for the coordinates to be given by distances, rather than their reciprocals.

A system, similar to the tangential, was suggested by Plücker[3]. He considers as the coordinates of a line (1) the tangent of the angle which the line makes with the negative X-axis, and (2) the intercept on this axis. This system also is not the dual of the Cartesian.

(d) Schwering and Franklin. A more exact plane dual[4] of the Cartesian was worked out independently by Schwering (1876) and Franklin (1878). In the system known as the Schwering system the coordinates of a line in a plane are determined as follows:-

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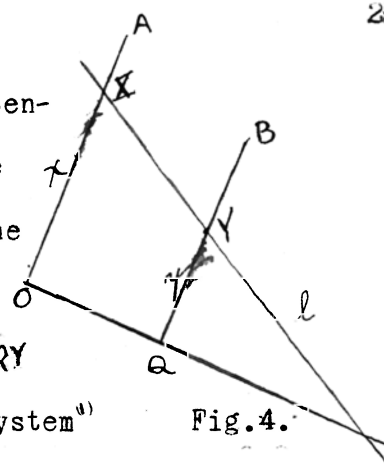
1) Jeffery. Quarterly Journal of Mathematics. IX (1869):189.

2) As to the possibility of an exact dual of the Cartesian, see Weinleitner Algebraische Kurven. 12-15.

3) Schwering. *Zeitschrift für Math. u.Physik XXI (1876):278-86.

4) Schlegel. Jahr.über die Fort.der Math. X (1878):439.
" *Zeitschrift für Math.und Phtsik XXIII(1878):195-6.



at the ends of a segment OQ draw two perpendicular lines OA and QB. Consider as the coordinates of a line l the lengths of the segments which it cuts off on OA and QB, respectively. In Fig.4. x = OX and y = QY are the coordinates of l. Schwering's system[1] is thus really only a dual of the rectangular Cartesian system.

Fig.4.

A more general system than Schwering's is the system known as the bipunctual, defined by Franklin[2]. This system is more general in that it allows the two parallel lines through O and Q to make any angle whatsoever with OQ. Such a system had, however, been used by Salmpnn at a time previous to that of either Schwering or Franklin, Salmon[3] mentioning such a system as early as 1869. He also gave some examples of its use, but did not note its most important property, viz. its duality to the Cartesian. In 1871, Unverzagt[4] also published an article in which this same system was independently established.

Casorati[5] has developed a system, somewhat similar to that given by Franklin, known as Casorati's coordinates; this, however also lacked complete duality to the Cartesian. This fact was pointed

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1) Schwering. Jahrbuch über die Fort. der Math. VIII(1876):414-5

2) Franklin. American Journal of Mathematics. I(1878):148-

3) Salmon. Conic Sections. 5th ed. 365.

4) Unverzagt. Jahrbuch über die Fort. der Math. III(1871):308-10

5) Casorati. *Nouvelles Analles. (2)XVII(1878):5-20.

by D'Arcais[1] and Casorati's system was modified to meet the additional requirements. As so modified, the system is identical with that given by Franklin and represents the general dual of the Cartesian system.

3. Perpendicular coordinates.[2] In the Cartesian system, the x cooddinate, say, of a point is determined by its distance from the X axis measured along a line parallel to the Y axis. In a perpendicular system, however, its distance is measured along a line perpendicular to the X axis, without reference to its relation to the other axis. Thus, in Fig.5. (x,y) are the oblique coordinates of the point P, and (x',y') are its perpendicular coordinates. The algebraic relations connecting the two systems are as follows:-

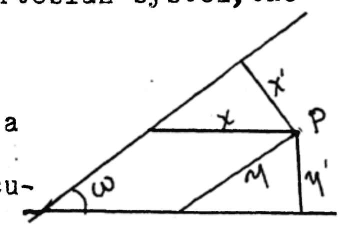

Fig.5.

$$\begin{cases} x' = x \sin \omega \\ y' = y \sin \omega \end{cases} \quad \text{or} \quad \begin{cases} x = x' \csc \omega \\ y = y' \csc \omega \end{cases}$$

where ω is the angle between the axes If $\omega = 90°$, the perpendicular system is identical with the Cartesian. It is evident that this system may be extended so that the coordinates of a point may be considered as its distances from the two axes, measured along lines parallel to any two given lines in the plane.

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1) D'Arcais. Jahrbuch über die Fort.der Math. X (1878): 441-2.

2) Whitworth. Trilinear Coordinates. 1-8.

4. Trilateral coordinates.** We will include under the term "trilateral" all systems in which the coordinates of a point are determined wholly or in part by its distances from three given lines[1]. All such coordinate systems are known as homogeneous since any algebraic curve expressed in terms of them gives rise to a homogeneous equation, although other definitions of homogeneous coordinates are given by

Heger.	Jahr. über die Fotr. der Math.	II(1869-70): 449-50.
"	*Zeitschrift für Math. und Physik.	XV (1870) 389-426.
Müller.	Encyklopädia der Math. Wissen.	III$_1$4 : 602.

(a) Möbius[2]. Areal. The first kind of trilateral coordinates to be used was the barycentric, or Schwerpunkt[3], invented by Möbius in 1837. Assuming three points of the plane as fundamental points, Mobius considered definite weights as being hung at each of the three points and then endeavored to locate the center of

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** German authors use the terms "trilinear", "trimetric" and "Dreiecks" instead of trilateral. The word "trilinear" has, however been universally used in the manner employed in this article. The word "triangular" has been variously used, being used as synonymous with the word "areal" by English writers. Moreover, the word "trilateral" seems more suggestive of the real character of the general system than trimetric, and is, therefore, used.

See	Fischer.	Koordinatensysteme.	50.
	Pascal.	Repertorium der Höheren Math.	II$_2$: 21.
1)	Klein	Einleitung in die höhere Geom.	I : 21-30.
2)	Möbius	Gesammelte Werke.	I: 50, 147.
3)	Bohren	Jahr. über die Fort. der Math.	XXXV 1904): 581,

gravity of the resulting system. In a brief review of Möbius' "Barycentric Calcul", in which the above method is given, Allardice[1] explains the plan of Möbius as follows:- " Let us confine our attention to the plane, and consider three points A, B, C. If weights a, b, c, be imagined at the points A, B, C, then this system will have a definite system of gravity, P say. In the determination of P only the ratios a:b:c are involved and there is a one-to-one correspondence between the position of P and the values of these ratios. The quantities a, b, c, or their ratios are therefore taken as the coordinates of P." Möbius also showed that in the system thus set up the coordinates (a,b,c) of

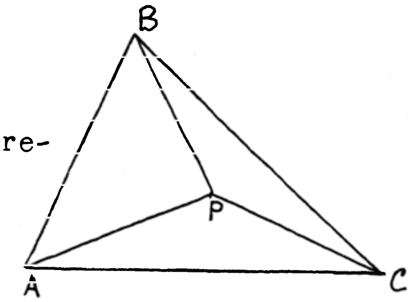

Fig.6.

the point P are proportional to the areas of the triangles PBC, PCA, PAB, as in Fig.6. For this reason this system is at present generally known as the areal[2] system. This term "areal" is, however, often applied only to the system in which the ratios of the three triangles to the original triangle are considered as coordinates;[3] this latter system being sometimes

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1) Allardice. Proc. Edin. Math. Society. X(1892):2-21.

2) Wolstemholme. Mathematical Problems 217ff.

3) For various disrinctions, and for various usages see the ff.

Ferrers.	Trilinear Coordinates.	99.
Milne.	Homogeneous Coordinates.	1-3.
Casey.	A Treatise on the Anal. Geom. etc.	64-5.
Whitwotth.	Trilinear Coordinates.	94-101.
Torrey.	Messenger of Math.	I(1862)220-9.
Bardelli.	Jahr. ü. der Fott. der Math.	VIII(1876)412-3.

sometimes distinguished from the areal by giving to it the name " triangular". Most authorities at the present time agree in calling the latter system tha actual areal, or absolute barycentric system, and use as the areal coordinates of a point P any values proportional to the three triangles PBC, PCA, PAB. The term "triangular coordinates" has also been used as denoting any values proportional to the areas of the three triangles. An important distinction between the actual areal areal coordinates of a point and its areal coordinates is that if (x,y,z) are the actual areal coordinates, these coordinates will satisfy the equation

$$x + y + z = 1$$

while the areal coordinates do not.

Guidice[1] gives the name "semi-homogeneous" to those systems of coordinates in which any form under consideration is expressable by a homogeneous equation in the coordinates, where the coordinates are subjected to the condition that a homogeneous function of them must be a constant. If this nomenclature were adopted, all trilateral systems would be called semi-homogeneous; we shall, however, continue to call them homogeneous systems.

(b) Plucker. Trilinear. Bobillier[2], in 1827, introduced a second kind of trilateral coordinates. Plucker[2], in 1829, completely developed this new system, but independently either of Bobillier

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1) Müller. Encyklopädie der Math. Wissen. III_1, 4:602, 641.

Guidice. Rend. Circ. mat. Palermo. XII (1898): 288ff.

2) Plücker Gesammelte Werke. I: 599; 124--58.

Merriman and Woodward Higher Mathematics 554.

or Möbius, and from considerations entirely different from those which lead Möbius to the invention of barycentric coordinates. Plucker obtained this same coordinate system while endeavoring to establish a more general coordinate system than any already known by him. According to his method let OO', OO'', O'O'' be three non-concurrent straight lines and let p, q, and r be the distances of any given point M from these three lines; see Fig.5. Consider as the coordinates of a point M the three numbers p, q, r, or numbers proportional to them. Assume that the signs connected with the distances are such that for any point within the triangle, p is positive, and both q and r negative. The system thus established is the same as that generally known at present as the <u>trilinear</u> system; the only difference being that at the present time the coordinates of any point within the triangle are assumed to be all positive[1]. A distinction is often made between the <u>actual</u> <u>trilinear</u>[2] system, in which the distances are taken as coordinates, and the <u>trilinear</u> system in which any set of numbers proportional to these distances are the coordinates are the coordinates of the point. The trilinear system is some-

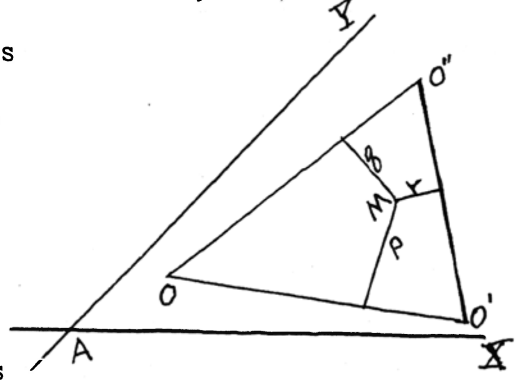

Fig.5.

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1) Casey. A Treatise on the Anal. Geom. of the etc. 61.

2) Milne. Homogeneous Coordinates. 6ff.

times referred to as the normal system. The actual trilinear coordinates of a point P $(\alpha\beta\gamma)$ satisfy the equation

$$a\alpha + b\beta + c\gamma = 2\Delta$$

where a, b, c are the lengths of the three sides of the triangle and Δ is the area of this triangle. The normal, or trilinear coordinates of a point will not in general satisfy this equation.

Salmon[1], in 1869, gave a different interpretation to the trilinear system. He showed that if α be placed equal to

$$x \cos \alpha + y \sin \alpha - p = 0 .$$

and β and γ equal to similar expressions, then any equation of the form

$$ax + mb + c = 0$$

can be thrown into the form

$$l\alpha + m\beta + n\gamma = 0 ,$$

provided the equations α, β, γ are linearly independent.

(c) Chasles. Projection. Chasles was much less definite in his statements than either Möbius or Plücker[2]; but he did have the concept of actual trilinear coordinates as is evidenced by a single paragraph (see Aperçu Historique des Methodes en Geometrie 632ff). His most important innovation, however, was the use of projection in determining the coordinates of a point; for example, as is shown in Fig. 6, he considered as the coordinates of the

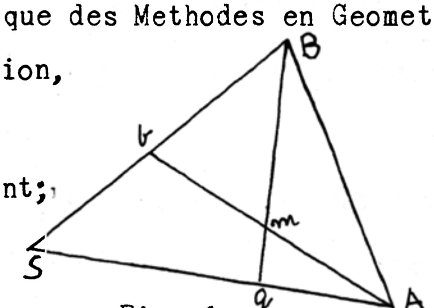

Fig. 6.

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1) Salmon. Conic Sections. 53-60.

2) Chasles. Geometrie Superieure. 315ff.

point m the numbers x and y where $x = \frac{aS}{aA}$

$y = \frac{bS}{bB}$

a and b being the projections of m upon the sides SA, SB respectively. Although, in general, he considered the point as given by two coordinates only, his system being in reality the ratio system extended to two dimensions, he did have the concept of homogeneous coordinates, as is evidenced by the paragraph referred to above. It is easily seen that it is a very short step from this special method of Chasles to the general projective system. This system is also known as the Chasles "Schnittverhältnis" system; it is mentioned in

Jahrbuch über die Fortschtitte der Mathematik. XXV(1893-4):1063.

where the reader is referred to

Zeitschrift für Math. und Physik. XXIV(1879):88.

*Nouvelles Annalles. (3) XII(1893): 360-87. 503-19.

These articles are not available to the writer.

(d) Hesse. Hesse[1] (1865) considered the special case in which one of the sides of the triangle became the ideal line, and thus obtained what is known as the homogeneous Cartesian, Hessian, or bilinear [2] system.

(e) Generalization. Scott[3] (1894) has shown that we may take as the coordinates of a point, multiples of its distances from the sides of a triangle, each distance, moreover, being multiplied

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1) Heffter u. Koehler. Lehrbuch der Anal. Geometrie. I:201-3.

2) Müller. Encyklopädie der Math. Wissen. III, 4 :645-7.

3) Scott. Modern Analytical Geometry. 8-9.

(~~veing multiplied~~) by a different constant. Thus x_1, x_2, x_3 may be considered as the coordinates of a point if they satisfy the equations

$$\rho x_1 = l\alpha$$
$$\rho x_2 = m\beta \qquad \rho \neq 0,$$
$$\rho x_3 = n\gamma$$

where α, β and γ are the trilinear coordinates of the point and $l, m,$ and n are any three constants different from zero**. This gives the general trilateral system; the previous systems being but special cases of it. The three axes of such a system having been fixed, there remain two degrees of freedom. There will be, therefore, in ordinary euclidean geometry, a doubly infinite number of trilateral systems; each corresponding to a definite position of a point in the plane as the unit point[1]. Thus the areal and trilinear systems are special cases of this general system, the centroid being the unit point in the areal system, and the incenter in the trilinear system. It is evident that only a few of the totality of such coordinate systems will ever be used; the value of this general system being that it makes it possible by selecting proper values for the multiplying constants, to make any particular point under discussion in a problem the unit point; thus simplifying greatly the work at hand. In other words, if our

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** The relation between coordinate systems and transformations is very close. See Newson Theory of Collineations.

1) Milne Homogeneous Coordinates. 11.

[on next page]
2) Grunert. *Archiv der Math. und Physik. LIII(1871):393.
Reum. Jahrbuch über die Fort. der Math. VII(1875):422-

triangle of reference is fixed, and any point E is chosen as the unit point the coordinates of any point P will be given by its distances from the three sides of the triangle ofreference, the disrances of E from the same sides being the units of measurements. For this reason a trilateral system is often known as a trimetric system[1], although trimetric is sometimes used as being synonymous with trilinear.

(f) Special systems. Two systems [1] closely related to the barycentric or areal coordinate system are the inertia system and Ceva's system, the latter being one in which the ratios of the reciprocals of the barycentric coordinates are regarded as the coordinates of the point.

Plücker[2], in 1829, introduced a non-homogeneous and non-linear system in which he considered as the coordinates of a point, two values μ and ν where $\mu = p:q$, $\nu = p:r$

p, q and r being the trilinear coordinates of the point under consideration. μ and ν are, thus, in reality merely the reciprocals of the linear coordinates of the point.

If we assume as our triangle of reference an equilateral triangle and consider its center as the unit point, we obtain the regular trilateral system[1].

If one vertex of the triangle of reference is assumed to be at infinity there arises the "Streifen".[3] angular, or

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1) Müller. Encyklopädie der Math. Wissen. III_1 4:645-7.

2) Plücker. Gesammelte wissen. Abhandlungen. I=127-9.

3) Fiedler. Die Darstellende Geometrie. III:91-2.

biangular[1] coordinate system. In this system, the coordinates of a point P are the cotangents of the two angles θ_1 and θ_2 which the lines joining P to two fixed points A_1 and A_2 make with the line A_1A_2, as in Fig.7. This system is a linear system only when θ_1 and θ_2 are measured in the same direction.

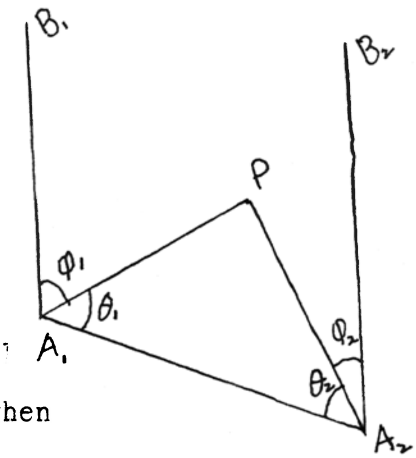

Fig.7.

A different system[2], closely allied to this, is the one in which the coordinates are assumed to be the cotangents of the angles which the vectors PA_1, PA_2 make with the parallel lines A_1B_1, A_2B_2 i.e. the cotangents of the angles φ_1 and φ_2 in Fig.7.

On account of its close relationship to trilateral coordinates we will consider at this time a non linear system. Let α, β and γ be the coordinates of a point. Consider as the new coordinates of the same point the values x, y and z, where

$$x = \frac{\beta}{\gamma}, \quad y = \frac{\gamma}{\alpha}, \qquad z = \frac{\alpha}{\beta} .$$

An advantage of this system is that the relation $xyz = 1$ which exists between the coordinates is independent of the size of the triangle of reference. These coordinates are called trigonal[3] coordinates. Trigonal coordinates of the m^{th} class

1) Biggin. Quarterly Journal of Mathemetics. XXV(1891):231-58.

2) Genese. " " " " XVIII(1882):150-4.

3) Levi. Jahrbuch über die Fort. der Math. VIII(1876):413-4.

may be obtained by means of the following equations, from a trilinear system $(\alpha\beta\gamma)$:

$$\frac{\beta}{\gamma} = x_1, \quad \frac{\gamma}{\alpha} = y_1, \quad \frac{\alpha}{\beta} = z_1,$$

$$\frac{y_1}{z_1} = x_2, \quad \frac{z_1}{x_1} = y_2, \quad \frac{x_1}{y_1} = z_2,$$

— — — — — — — — — — — — —

— — — — — — — — — — — — —

$$\frac{y_{m-1}}{z_{m-1}} = x_m, \quad \frac{z_{m-1}}{x_{m-1}} = y_m, \quad \frac{x_{m-1}}{y_{m-1}} = z_m.$$

It should be noted, however, that some of the above systems are not linear.

Regular trilateral coordinates are used in chemistry [1] for representing the concentrations of the components of a chemical system; and also in problems [2] in physics and engineering for plotting the curves of the properties of alloys made up of three metals, mortars with graded sizes of sand, etc. A very interesting application of the regular trilateral system is the geometric solution [3] of cubic equations with three real roots. As so used, it is sometimes called the polygonal system. It is worthy of note that the idea which lead to the first concept of any kind of trilateral coordinates came from a physical problem, i. e. to determine the center of gravity of weights at the vertices of a triangle.

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1) Findley. Phase Rule. 237–42.

2) Taylor and Thompson. Concrete, Plain and Reinforced (1906): 143.
Johnson. The Material of Construction. (1908): 175.

3) Sautreaux. *Memoires Couronnes et autres Mem. etc. XXXVI (1883)

4. Tangential. We will include under the general term "tangential" all those systems of line coordinates in which the coordinates of a line are determined either wholly or in part by its distances from three given points. Other terms which are at times applied to this system are "Dreipunkt" and "Three point tangential"[1]. If we consider as the equation of a line the general form

$$u_1x_1+u_2x_2 + u_3x_3 =0$$

where x_1, x_2, x_3 are variables in some system of trilateral coordinates, we may consider the numbers u_1, u_2, u_3 or their ratios as the tangential coordinates of the line[2]. This method of introducing tangential coordinates is that given by Plücker, the first to use them. Chasles [3] also took up the consideration of tangantial coordinates, but from a more strictly geometric point of view. He assumes as the coordinates of a line its distances from the three vertices of a triangle. It is easy to see that we are able by dualization to determine the plane duals of the various trilateral systems; the plane dual e.g. of the actual trilinear system being the tangential system in which the distances of the line from the three vertices are taken as coordinates.

6. Simultaneous point and line systems.[4] It is desirable to associate point and line coordinate systems in such a way that

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1) Jeffery. Quarterly Journal of ~~Science~~ Math. IX(1868):1ff.

2) Salmon. Conic Sections. 65.

3) Plücker. Gesammelte wissen. Abhandlungen. I:127-9.

4) Milne. Homogeneous Coordinates. 63ff.

that we can use both simultaneously. If this is to be done, it is evident that there must be some relation connecting, say, the trilateral coordinates (x_1, x_2, x_3) and the tangential coordinates (u_1, u_2, u_3). The simplest general relation, and the only one considered is the following

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 ;$$

if we assume that the coordinates (x_1, x_2, x_3) of a point and the coordinates (u_1, u_2, u_3) of a line through it satisfy this relation we call the systems "simultaneous", and call the equation $u_1x_1 + u_2x_2 + u_3x_3 = 0$ the combined equation of the point and line. It should be emphasized that in such simultaneous systems, if (x_1, x_2, x_3) are taken as areal point coordinates, (u_1, u_2, u_3) will be areal line coordinates, whereas if we consider (x_1, x_2, x_3) as trilinear point coordinates, (u_1, u_2, u_3) must be taken as trilinear line coordinates. It should also be noticed that in general two systems which form a simultaneous point and line system are not the duals of each other; e.g. the plane dual of the trilinear system is the areal tangential system [1].

7. Multilinear. Plücker [2] (1829) was the first to see the possibility of using four or more lines as lines of reference in a coordinate system He states that the position of a point may be given by the distances p, q, r, s, t, -- from certain definite lines in the plane, and that the equation of a curve expressed in terms

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1) Scott Modern Analytical Geometry. 12-3.

2) Plücker Gesammelte wissen. Abhandlumgen. I:153-4.

of the variables of such a system, is of the form $F(p,q,r,s,t\cdots)=0$ where, moreover, $s, t, \cdots$ are expressable as linear functions of p, q and r. He also showed that the degree of this equation $F=0$ is the same as the order of the curve which it repeesents. This equation is homogeneous, but may, of course, be made non homogeneous by dividing through by any one of the variables in it and then considering the ratios of these variables as the new variables. It was not, however, until 1835 that Plucker[1] saw the advisability of limiting these lines of reference to four. In such a system, generally known as a <u>quadrilinear</u> system, there is no peculiarity attached to the line at infinity.

Quadrilinear coordinates have been studied extensively by English writers. Whitworth[2] distinguished between <u>perpendicular quadrilinear</u> coordinates $(\alpha, \beta, \gamma, \delta)$ of a point, which are the distances of the point from the four lines, and the <u>quadrilinear</u> coordinates $(\alpha\beta\gamma\delta)$ which are certain multiples of $(\alpha\ \beta\ \gamma\ \delta)$. It is at once evident that two relations must connect any such set of quadrilinear coordinates, ther being but two degrees of freedom in the plane.

Multilinear coordinates have been called by Morley[3] "<u>super-numerary</u> <u>homogeneous</u> " coordinates, or <u>symmetrical</u> coordinates.

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1) Plucker. *Systeme der anal.Geom. (1835).

2) Whitworth. Messenger of Mathematics. I(1862):193ff.
" Trilinear Coordinates. 306-23.

3) Morley. Transactions of the Amer.Math.Soc.IV(1903):288-96.

In fact all those line systems in which more than three lines are used as lines of reference, are included under the general term "supernumerary linear point coordinate system", the general name for them is, however, "polygonal" coordinates[1].

The dual systems in which the coordinates of a line are regarded its distances from four or more fixed points do not seem to have been used.

8. Projective systems [2]. The idea of projective coordinates originated with Möbius[3]. Ten years later, Chasles developed a similar system in three dimensions which is so simple of application to two dimensions that very probably it was known to Chasles[4]. The two dimensional system will be developed here. Given a fixed triangle ABC, see Fig.8; fix on two of its sides, say AB BC two unit points E_1, E_2 respt. Let P be any point in the plane. Project it from the vertices C, A onto the sides AB, BC respt., calling these projections P_1, P_2 respectively.

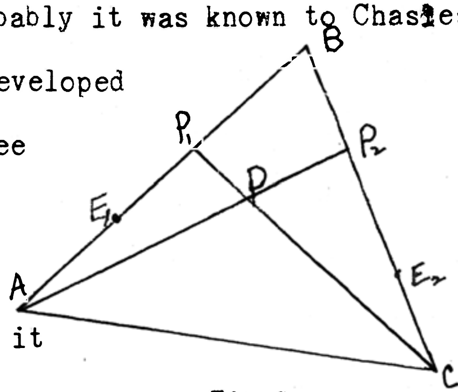

Fig.8.

Consider as the coordinates of the point P the double ratios of

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1) Chasles. Traite de Geometrie Superieure. 290-300.
Klein. Vorlesungen uber die Ikosaeder. 162-3.
Maatz. *Zur Geschichte der Polyeder Koordinaten.
2) Fischer. Koordinatensysteme. 48-71.
Hime. Anharmonic Coordinates.
3) Möbius. Gesammelte Werke. I:219-36, 291-308.
4) Müller. Encyklopädie der Math. Wissen. III, 4 :640n.
Chasles Geometrie Superieure. 341ff.

($ABEP_1$),($BCEP_2$) respt. This system differs but little from the ordinary projective system, and in three dimensional space, at least, is known as the Chasles system.

The real development of the idea of projective coordinates is due to von Staudt, Hamilton and Fiedler. von Staudt's article[1] was not available to the writer, but his general method follows[2]. Consider two pencils of lines and establish coordinate systems on them, the line joining the vertices of the two pencils being self corresponding. It is evident that one line of each pencil will pass through any point; associate with any point as coordinates, the values of the coordinates of the lines of the two pencils passing through it. This system at first was not applicable to the points on the junction line of the two pencils, but the system was later extended to include such points.

Hamilton[3] assumed a triangle, see Fig.9, as given and fixed a unit point O. Considering P as an arbitrary point, he projects O and P from the vertices A, B, C onto the sides BC, CA, AB, respt. In this way he obtained upon each side of the triangle, for any point P, four definite points.

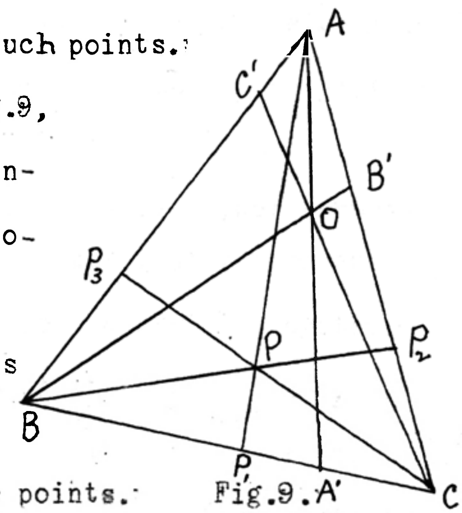

Fig.9.

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1) von Staudt *Beitrage zu Geom.d.Lage. (1857).

2) Visnya Jahr.uber die Fort.der Math. XXXVII(1906)585.
Archiv der Math u. Physik. (3) X(1906)337-9.

3) Hamilton. *Nat.Hist.Review and Q.J.of Science(1860) 242- 325- 506-
Elements of Quaternions. I:23-9.

He showed that it is possible to express of the four points on each side of the triangle as follows:-

$$(BA'CP) = \frac{y}{x}, \quad (CB'AP_2) = \frac{z}{x}, \quad (AC'P_aL) = \frac{x}{y}.$$

In his own words," Any two of these pencils suffice to determine the point P when the triangle ABC and the origin O are given and therefore it appears that the three coefficients x,y,z or ann scalars proportional to them,may be conveniently called the Anharmonic Coordinates of the point P with respect to the given triangle and origin". He also showed that a point moving along a line l must satisfy an equation of the type

$$lx + mn + nz = 0$$

and then calls l,m,n the anharmonic coordinates of the line l. He,however,stated that there is no connection between the geometric interpretation of the trilinear system,and such a projective system.

Fiedler[1] followed the general method of Hamilton,but showed that the different systems of trilateral coordinates were but special types of this projective system;any particular system depending upon the particular position of the unit point O,e.g. if P be the center of gravity of the triangle,the coordinates (x,y,z) are areal coordinates,or as Fiedler calls them,surface coordinates.

Especial attention is here called to the "Schnittverhältnis"

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1) Fiedler Die Darstellende Geometrie. III:73-6.
" *Viertel.Natur.Gesell.zu Zurich. XV(1870)169.

system noted on page 36, its similarity to the projective system being very marked.

Special systems.[1] The projective system defined above in which the three sides of the triangle of reference are used, gives rise to homogeneous equations. If, howeever, the cross ratios on only two sides of the triangle of reference are used, the corresponding homogeneous equations will be non-homogeneous and the system used will therefore be called non-homogeneous.

If one vertex of the triangle of reference be projected to infinity, we obtain the "Streifen" system[1] which may be either homogeneous or non homogeneous.

Dual. Projective line coordinates are obtained in a dual manner by considering as the fundamental elements of the system the vertices of a triangle of reference and a unit line.

It is evident that all systems which have been developed for the points in a plane are immediately applicable by projection[2] to the lines on a point, and all point systems which have been dualized for lines in a plane are immediately applicable by projection to the planes on a point.

9. An algebraic system.[3] Consider the equation

$$am^2 + bm + c = 0$$

where m is a parameter. Then, as has been shown, to each value of

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1) For a number of unimportant variations in projective coordinates see Fiedler. Die Darstellende Geometrie. III:115,91-2.

2) Veblen and Young. Projective Geometry. I:

3) Darboux. Jahr. uber die Fort. der Math. IV(1872)319-20.

$\underline{m}$ there corresponds a tangent to the conic $b^2 - 4ac = 0$

It is evident that if a tangent is to pass through a point $P'(a', b', c')$, $\underline{m}$ must satisfy the equation

$$a'm^2 + b'm + c' = 0$$

and must, therefore be equal to one of the two roots $\underline{\rho_1}, \underline{\rho_2}$ of this equation, These two numbers $\underline{\rho_1, \rho_2}$ are connected with the two values $a':b':c'$ by the equation

$$\frac{a'}{1} = \frac{b'}{(\rho_1 + \rho_2)} = \frac{c'}{\rho_2 \rho_1}$$

and may be considered as the coordinates of the point $\underline{P'}$. This system is intimately connected with the projective one dimensional system considered on page 2.

It has been shown that a homogeneous equation of degree $\underline{n}$ in $\underline{\rho_1}$ and $\underline{\rho_2}$ represents a curve of degree $\underline{n}$; but that a non-homogeneous equation of degree $\underline{n_1}$ in $\underline{\rho_1}$ and $\underline{n_2}$ in $\underline{\rho_2}$ represents a curve of degree $(n_1 + n_2)$. Applying this to the case $n = 1$, the conclusion is that this system is a linear one.

B. NON LINEAR PROJECTIVE COORDINATES.[1]

If we assume two pencils of lines in the plane, as fixed, it is evident that any point in the plane will be definitely determined if we are given the line of each pencil passing through it. Futhermore it is evident, that if we establish on each pencil of lines a projective coordinate system, the coordinate of a point

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1) Müller Ency. der Math. Wissen. III, 4 : 654-6.
also cf. Kasner Trans. of the Amer.Math.Soc.I(1900): 430 ff.

may be taken as those of the two lines passing through it. Such a system will be in general a non-linear projective coordinate system, for in general they determine a point uniquely, and all points whose coordinates satisfy a linear equation lie on a conic section passing through the vertices of the two pencils and through the intersection of the lines which have the coordinate ∞. If the line joining the two vertices has ∞ as its coordinate when considered as belonging to either pencil, i.e. if it is a self corresponding line, the above non-linear system will reduce to a linear system. The general linear projective coordinate system is, therefore, a special case of the non-linear one.

If the two vertices of the pencils are the two absolute points of the plane, the resulting system is called non-linear minimal system.

C. POLAR SYSTEMS.

1. Polar coordinates. This system in plane geometry was the second one to be developed, and while it does not have as general an application as the Cartesian system, still it simplifies many problems in analytic geometry and calculus. Cartesian coordinates and these polar coordinates are the only ones referred to in the majority of books on plane analytic geometry and calculus. In this system the position of a point in a plane is determined by (a) its distance from a fixed point, called the pole, or origin, and (b) the angle which the line joining the point to the pole makes with a fixed line passing through the pole,

the fixed line being called the initial line[1], angular axis[2], or prime radius[3]. The line joining the point to the pole is variously known as the radius vector[4], "Leitstrahl" or directional line[5], "Fahrstrahl"[5], radiant[6], and polar axis[7]; and the angle between the radius vector and the initial line as the vectorial[4], variable[2], or directional[7] angle. Its relation to the rectangular system is simple, thus, if (ρ,θ) and (x,y) are the polar coordinates and the rectangular coordinates respectively of a point, they are connected by the following equations

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases} \quad \text{or} \quad \begin{cases} \rho^2 = x^2 + y^2 \\ \theta \equiv \arctan y/x \end{cases}$$

As has been mentioned there is some question as to the inventor of polar coordinates. The credit is, however, generally given to Jacob Bernoulli[5]. In spite of their comparatively early origin, they were not generally known during the Eighteenth century; the equation of a straight line in polar coordinates not being given

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1) Bailey and Woods. Analytic Geometry. 13.
2) Coffin. Conic Sections and Analytic Geom. 84.
3) Ferrers. Quar. Jour. of Math. I(1857):210.
4) Ashton. Analytic Geometry. 63,73.
5) Muller. Ency. der Math. Wissen. III_1 4:656-8.
6) Hutton. A Course in Math. (1843) II:247.
7) Nichols. Analytic Geom. 4-5.
* ——— *Transactions of the Roy.Soc.Edin. XII :408.

[next page]

before 1830[1]. Polar coordinates were used but little, if at all ni analytic geometry in 1813[2], and rarely in calculus before 1845[3].

The polar system is a curvilinear system, as are all special and allied forms of polar coordinates. These forms of polar coordinates will, however, be considered at this time.

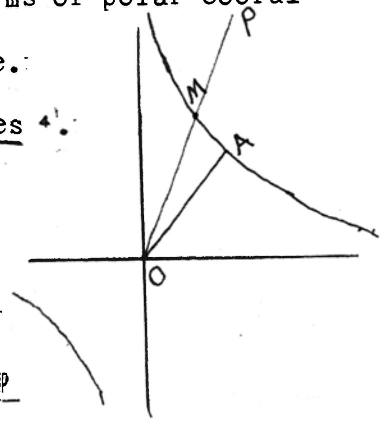

Fig.10.

(a) Elliptic and hyperbolic polar coordinates[4]. Consider an equilateral hyperbola, with semi-transverse axis equal to unity. In Fig. 10 let a ray OP cut the hyperbola in a point M. Let $r = OM/OA$, $\varphi = 2\angle AOM$. The values r and φ are known as the "Polar coordinates of an equilateral hyperbola". In a similar manner, any ellipse or hyperbola may be used, obtaining thus the elliptic polar and hyperbolic polar coordinate systems. These coordinates are also known as Laisant's coordinates, in honor of their inventor Laisant.

(b) Lemniscate polar coordinates[5]. This system is used in the "Abbildung" from the z to the w plane in the theory of complex variables. Its name comes from the fact that to a sheaf of concentric circles about the origin in the w plane, there corresponds a sheaf of confocal ~~ellipses~~ lemniscates in the z plane, provided the "Abbild

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2) Hutton. A Course in Mathematics. (1813) III.

3) Davies. Philosophical Magazine. (3) XXI(1842)190–

4) Muller. Encyklo. der Math. Wissen. III_1 4:658–60.

Laisant. *Laisant Essai sur les fonct. hyper. Paris (1874).

5) Schulz. Jahr. uber die Fort. der Math. XX(1888):685–6.

ung is given by the relation

$$w = z^n + a_1 z^{n-1} + \cdots + a_n \cdot$$

(c) Polar directrix coordinates.[1] Consider as the coordinates of a point (a) its distance from a fixed point, and (b) its distance from a fixed line. These distances determine its polar directrix coordinates[2]. Each pair of values will, in general, determine either no point or two points. The characteristic properties of this system may be obtained from the theorems in conic sections.

2. Duals of the polar system.

(a) Ferrers and Weinmeister. Ferrers[2] was the first to define a system dual to the polar system; calling it the tangential polar system. He defines as the tangential polar coordinates of a line; (a) the perpendicular distance of that line from a given point, and (b) the angle which this perpendicular makes with a fixed line through the point, or pole. This same system was later independently established by Weinmeister[3] who called it the polar line system.

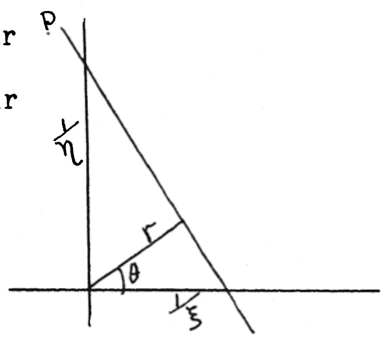

Fig.11.

A very simple relation exists between this system and the Boothian tangential system as can easily be seen from Fig.11 where

$$\xi = \frac{\cos\theta}{r}$$

$$\eta = \frac{\sin\theta}{r}.$$

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1) Niewenglowski Cours de Geom.Analytique. I:31.

2) Ferrers, Quarterly Jour.of Math. I(1857)210-8.
" Trilinear Coordinates. 143.

3) Weinmeister Jahr. über die Fort. der Math.VIII(1876)415-6.
" *Zeitschrift für Math.und Physik XXI(1876)301-24.

(ξ, η) being the Boothian tangential coordinates of the line p, and (r,θ) its polar line coordinates.

(b) Schlegel. To Schlegel, however, is given the credit[1] for inventing the system dual to the polar. His own article[2] is not available, but in making a review of D'Ocagnes[3] book upon axial coordinates, Schlegel states[4] that they are the same as his <u>reciprocal polar</u> coordinates. This leads us to consider the <u>axial</u> system more closely. In this system[5] a straight line is determined by the angle φ which it makes with a fixed axis and by the abscissa p of its intersection with this axis with respect to a fixed point upon it. These are its axial or reciprocal polar coordinates. Casey[6], however, had previously developed this system and used it extensively, though perhaps (the original articles are not available) without making any note of its dual relation to the polar. The relation between the polar line system (r,θ) and the reciprocal polar system (ρ,φ) is given by the following equations, as in Fig.12.

$$\rho = r\cos\theta,$$
$$\varphi = \theta + 90^\circ$$

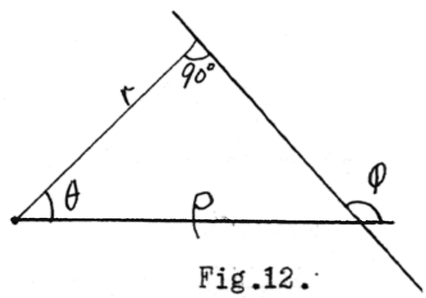

Fig.12.

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1) Müller. Math. Vocabularium 160.
2) Schlegel. *Association Fr.p.l'avan. sciences nat.Con. Grenoble.
3) D'Ocagne. *Coordonnees paralles et axiales. Paris 91s.
4) " Jahr.über die Fort.der Math. XXVII(1885):675.
5) " " " " " " XVI(1884):614-5.
" *Nouvelles Annalles. (3) III)1884):545-63.
6) Casey. Jahr.uber die Fort.der Math. IX(1877):473-4.

The supplement of φ has at times been considered as one coordinate instead of φ itself[1].

(c) Balitrand[2]. A well known form of the equation of a straight line is

$$x \cos \varphi + y \sin \varphi = p \quad .$$

Balitrand takes as the coordinates of such a line the values φ and p, he develops this system to some length, but does not give it a distinctive name.

(d) Unverzagt[3]. As in Fig 13, let two lines O_1A and O_2B be drawn through two fixed points O_1 and O_2. Let a line l meet the axis O_1O_2 at an angle φ and intersect the two lines O_1A and O_2B at X and Y respectively. Let the cross lines O_1Y and O_2X intersect at M. Let u be the length of the segment through M parallel to O_2Y lying between l and O_1O_2. Then the values of u and φ may be used to determine the coordinates of l.

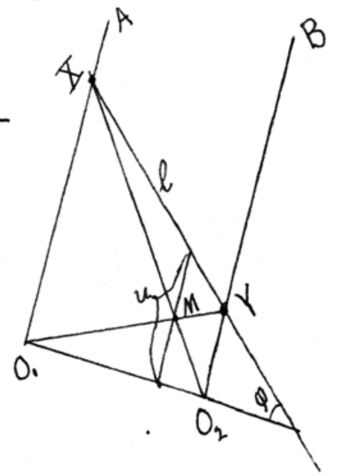

Fig.13.

3. Bipolar coordinates. The bipolar system of coordinates is due to Grunert (1859)[4] although Salmon (1852)[5], and perhaps others, had previously been aware of the principles involved. In this system, the coordinates of a point are its distances from two fixed points, or poles. The system is variously known as the

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1) Casey. *Phil.trans. Rohal Soc.of London.CLXVII(1877)367-440.
2) Balitrand. *Nouvelles Analles. (3) XII1893)256-8.
3) Unverzagt Jahr.uber die Fort.der Math. III (1871):308-10.
4) Grunert *Archiv der Math.und Physik. XXXII(1859):444-69. however see Cayley. Collected Works. III:258-61.
5) Salmon, Higher Plane Curves. (1873ed): 258-61

(a):	bipolar	Johnson.	Jahr. der Fort.Math.	VII(1875)448.
(b):	dipolar	Scott.	Mod.Analytic Geom.	1.
(c):	biradial	Casey.	A Treat.on the Anal.Geom.etc.	18-9.
(d):	vectorial	Bassett.	Elem.Treat.Cubic and Quar.Curves	173.
(e):	bivectorial	Niewenglowski.	Cours de Geom.Anal.	I:30-1.

In general to each pair of values there correspond two points, situated symmetrically with respect to the polar axis joining the two points, as in Fig.14. Accordingly it is impossible to obtain the equation of a straight line without including the line symmetric to it with respect to the polar axis. However, the equations of some important curves may be expressed very simply in these coordinates [1].

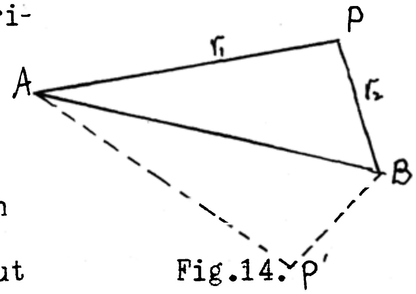

Fig.14.

Allied systems. The following systems are closely related to the bipolar system.

(a) The circular system [2] in which, if r_1 and r_2 are the distances of a point P from two fixed points, A and B, see Fig.14, we define as the circular coordinates of P the values ρ and θ, where $\rho = \frac{r_1}{r_2}$, $\theta = \angle BPA$. This system may be extended, just as the ratio system was extended to the cross ratio system, and a semi-projective may thus be obtained.

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1) de Vries. Jahr.uber die Fort.der Math. XXVII(1896):491.
*Akademie van Wetenschappen. IV(1896):219-24.

2) Johnson. *The Analyst. X(1883):129-35.

(b): A special type of confocal cyclic curve system [1] in which we adopt as the coordinates the values u,v where

$$u = \frac{r_1 + r_2}{2} \quad , \quad v = \frac{r_1 + r_2}{2}$$

r_1 and r_2 being the bipolar coordinates of the point; a special type of this system has been discussed by Heine, and the name "elliptic" assigned to it by him.

(c): A system [2] in which the numbers r_1^2, r_2^2 are considered as the coordinates; this also being a special bicyclic system.

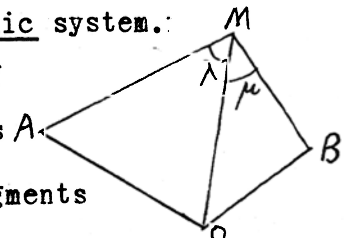
Fig.15.

(d): A system [3] in which the coordinates of a point M are considered as being the angles λ, μ which are subtended at it by the two segments OA and OB; see Fig.15. In this system λ = a const. gives a circle; a point thus being given as the intersection of two circles.

(e): A system [4] very similar to (d), differing from it only in that three nonconcurrent lines are used instead of two, and the three trigonic coordinates (as they are called) of a point being taken as the angles subtended at it by the three sides of the triangle of reference. This is a homogeneous system corresponding to the non-homogeneous system (d), which was invented later, and, probably, independently.

4. Tripolar coordinates. There are two general types of tripolar coordinates.

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1) Siebeck. *Journal für Math. LVII(1860): 359-
" LIX(1861)173-

2) Müller. Encyklopädie der Math.Wissen. III_1 4:687,670.

3) Habich. Jahr.über die Fort.der Math. XVI1884): 615-6.
" *Nouvelles Annalles. (3)III(1884)353-67.

4) Walton. Quar. Jour of Math. IX(1868)340-3.

(a): The coordinates of a point are associated with the distances of this point from three fixed points. Poulain [1] assumed that these three points were the vertices of a triangle, whereas de Vries considered these three points as being collinear[2].

(b): The coordinates of a point are considered as being the squares of its distances from three fixed non-collinear points.[3]

5. Quadripolar coordinates. Quadripolar coordinates have been developed by de Vries [4] who defined them as the distances of a point from the four vertices of a quadrangle. He also developed the special case in which two of the vertices lie symmetrically upon the perpendicular bisector of the line joining the other two.

6. Multipolar coordinates. The system of multipolar coordinates [4], in which the coordinates of a point are given by its distances from three or more points has been used by de Vries.

A system [5] similar to the multipolar is that in which the n coordinates of a point are the n tangential distances of the point from n given curves.

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1) Poulain. Jahrbuch über die Fortschritte der Math. XX[illegible]I 491.
" " " " " " " XXIII 720.
" *Journal de Math. elemen. etc. (3) III (1889) 3-10 etc.
*Journal de Math. speciales etc. (3) V (1891) 265-76.

2) de Vries. *Archives du Musee Teyler (2e serie) V (1898).

3) Lucas. Jahr. über die Fort. der Math. XXI (1889) 680.
*Mathesis. Receuil Math. IX (1889) 129-34, 173-81.
Casey. A Treatise on the Anal. Geom. etc. 309.

4) de Vries. Jahr. über die Fort. der Math. XXVII (1896) 491-2.

5) D'Ocagne. " " " " " " XXIX (1898) 489-90.
" *Nouvelles Annalles (3) XVII (1898) 115-8.

7. Physical polar systems.

(a) Dipolar coordinates were introduced by Neumann [1] in the mathematical theory of electricity. Using the upper half of the complex plane, he places

$$\xi + i\eta = a\frac{1 + e^{\theta + i\omega}}{1 - e^{\theta + i\omega}}$$

where ξ, η are the rectangular coordinates of the point, and a and e are constants. He calls the values θ and ω the dipolar coordinates of the point.

(b) Peripolar coordinates were also introduced by Neumann [2], being also used in the mathematical theory of electricity.

D. CURVILINEAR SYSTEMS.

The early historical development of curvilinear coordinates has been noted. The first general treatment of these coordinates was by Lame. He considered primarily the space of three dimensions, and regarded curvilinear coordinates in two dimensions merely as special cases.

A curvilinear coordinate system is simply any system in which a point is given by means of the intersection of two curves. If these belong to two families of curves, each having a single parameter, then it is evident that the coordinates of the point are the values of the parameters which correspond to the two curves passing through the point.

Curvilinear coordinates [3] may also be established from an algebraic point of view. Consider any two independent functions

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[1] Neumann Mathematische Annalen. XVIII(1881)196ff.

[2] Müller Math. Vocabularium. 160.

[3] " Encyklopädie der Math. Wissen. III_1, 4:629-44.

$f_1(x,y,x',y')$, $f_2(x,y,x',y')$ continuous in x and y and containing two parameters x' and y'. Assume further that $f_1 = 0$ and $f_2 = 0$ can be solved for x' and y' giving

$$x' = F_1(x,y)$$
$$y' = F_2(x,y)$$

The equations $F_1 = x'$, $F_2 = y'$ represent for any definite values of x' and y', two definite curves. The point or points in which the two curves intersect may then be considered as having as coordinates, the values x', y', and to them we give the name curvilinear coordinates.

If the two families of curves obtained by giving to x' and y' all possible values are such that each curve of one family meets all the curves of the other family at right angles, then the system determined by these two families of curves is called an orthogonal system[1]. The rectangular Cartesian system and the polar system are examples of orthogonal curvilinear systems. It is thus evident that in the broadest sense, the curvilinear system includes all point coordinate systems, or in other words, is the general point system.

From the standpoint of function theory this system is known as the generalized system [2] and is used in physics. This system was first used by Lagrange in studying the equations of motion, and is sometimes known as the Lagrangian system.

Homogeneous curvilinear coordinates may be obtained by considering three curves, rather than two.

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1) cf. Darboux. Lecons sur les Systemes Orthogonaux.
2) Webster. The Theory of Elect. and Magnetism. 116.

Elliptic coordinates.[1] A special type of curvilinear system when the two families of curves are taken as a family of confocal ellipses, and a family of confocal hyperbolas, both families having the same focii. It can be shown that the elliptic system thus obtained is also an orthogonal system; consider the equation

$$\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1 \quad , \quad a \geq b \ , \quad \lambda < a \ .$$

To definite values x', y' there will correspond definite values of λ_1, λ_2, the roots of the above equation; these values $(\lambda_1 \lambda_2)$ being called the elliptic coordinates of the point P in question. Geometrically it may be shown that the two values of λ determine an ellipse and an hyperbola which pass through the point. As is evident, every pair of values $(\lambda_1 \lambda_2)$ determines, in general, four points, one in each quadrant.

Parabolic coordinates.[2] These are obtained by a method similar to that which gave the elliptic system; in this case, however, the equation

$$\frac{y^2}{b-\lambda} + 2x + \lambda = 0$$

is used to determine the two families of confocal parabolas, the two families having the same focus.

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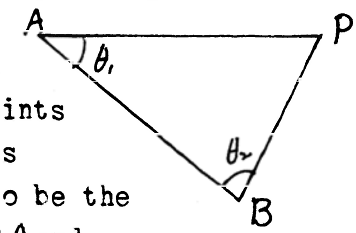
1) Staude. Jahr. über die Fort. der Math. XIII(1881): 523.
" Focal eigen. der Flachen etc. 161-5.
also cf. Klein Einleitung in die höhere Geom. I: 50-71.

2) Staude. Focal eigen. der flachen 2nd Ordnung. 171-3.
For special kinds of parabolic coordinates see

Baer. *Parabolische Coordinaten in der Ebene etc.

E. SPECIAL SYSTEMS.

In this section will be considered some of the special systems by means of which a point in a plane may be fixed. The majority of them are not often used, and the following arrangement is a mere casual one.



(a) Biangular.[1] In this system the angles which the radii vectors from two fixed points to the given point make with with the axis joining the two fixed points are assumed to be the coordinates of the point. Thus in Fig.16 θ_1 and θ_2 are the biangular coordinates of the point.

Fig.16

This a non linear system entirely distinct from the linear system in which cot θ_1 and cot θ_2 were considered as the biangular coordinates of the point P.

(b) Intrinsic.[2] Consider any given curve and fix on it a definite point as the origin. It is possible to obtain a relation between the distance s, from the origin, of any point on the curve, measured along the curve, and the radius of curvature R at the point in question. This relation is independent of the position of the coordinate axes. The values s and R so obtained are, therefore, called the intrinsic or natural[3] coordinates of the point.

(c) Semi-natural.[4] Any system in which either the length of an arc from a given point or the value of the radius of curvature at the point determines one of the coordinates of the point is called a semi-natural system. One form of such a system is that in which the rectangular Cartesian coordinate x and the length of arc s determine the coordinates of the point. Such coordinates (s,x) are called "Bogen" coordinates.

1.	Casey	A Treatise on the Anal.Geom. etc		
2.	Murray	Differential snd Integral Calculus		483 6.
	Walton	Quarterly Jout.of Math.	V(1862)	260 4.
3.	Weinlieter	Specielle ebene Kurven		169 81.
4.	"	" " "		392.

(d) Vectorial.[1] Here the coordinates of a vector in space are assumed to be two vectorial quantities defined as follows. Draw through a point O taken as the origin, a vector OD parallel to the given vector AB and equal to it in length, and a vector OM perpendicular to the plane OAB at O and equal to the moment of the vector AB upon this plane. The two vectors OD and OM thus obtained may be regarded as the vectorial coordinates of the vector AB.

(e) Bipunctual.[2] Assume two fixed points A and B, and two fixed lines a and b. Join any arbitrary point P to the point A and draw through B a line parallel to b cutting PA in B', as in Fig.17. Likewise join P to the point B and draw through A a line parallel to a, cutting PB in A'. The lengths AA' and BB' may be taken as the coordinates of the point P. These coordinates are known as bipunctual point coordinates.

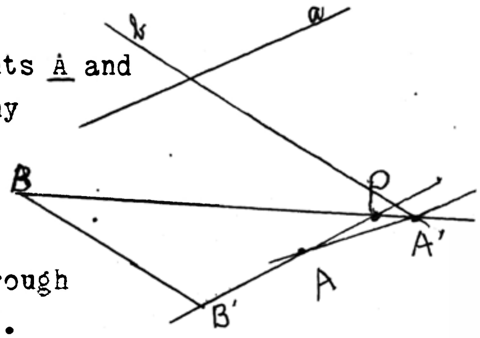


Fig.17.

(f) Logarithmic.[3] It is sometimes advisable to consider as coordinates the logarithms of the Cartesian coordinates instead of the numbers themselves. These are known as the logarithmic coordinates of the point.

(g) Tricyclic.[4] Consider three circles with non-collinear centers. New coordinates of a point P, with rectangular coordinates (x,y) may be taken as equal to the three values obtained by the substitution of its (x,y) coordinates in the rectangular equations of the three given circles. These coordinates are known as tricyclic coordinates and were first employed by Casey in 1869.

(h) Bicyclic.[5] The above tricyclic system is homogeneous, the corresponding non homogeneous system being that in which two rather than three circles are used, and is known as the bicyclic system.

(i) Circular. If x and y are the rectangular coordinates of

1. Niewenglowski Cours de Geom. Anal. III65 6.
2. ---------International Encyclopedia atr."Coordinates".
3. Vincent Jahrb.uber die Fott.der Math.XXIX(1898)484.
   " Raport of the Brit. Association for the etc.1898
4. Casey A Treatise on the Anal.Geom.of the etc. 301.
5. Muller Encyklopadie der Math. Wissen. III 4:687.

a point in the complex plane, the values

$$x + iy = X$$
$$x - iy = Y$$

are uniquely determined by them. It is sometimes advantageous to use these new values (X, Y) as the coordinates of the point. They are then called <u>circular</u> coordinates.[1]

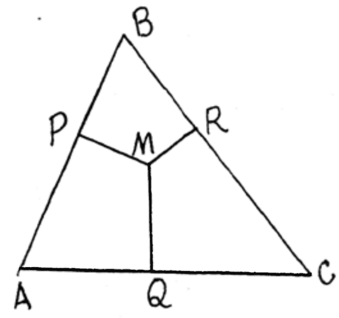


Fig. 18.

(j) <u>Sub-linear</u>. Let ABC be a fixed triangle, and <u>M</u> any point in the plane. Drop perpendiculars MP, MQ, MR upon AB, AC, BC respt., as in Fig. 18. The three distances AP, BR, CQ (or PB, RC, QA) may be considered as the coordinates of the point <u>M</u>. These coordinates are known as <u>sublinear</u> or <u>sub-trilinear</u>.

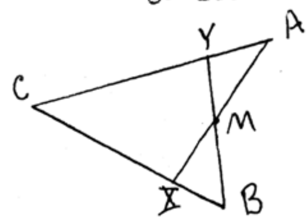


Fig. 19.

(k) <u>Lemoine</u>. Fix three points <u>A</u>, <u>B</u>, and <u>C</u>; draw the lines CA and CB. Let <u>M</u> be any point in the plane; join it to <u>A</u> and <u>B</u> and let the lines MA and MB meet CB adnd CA in the points <u>X</u> and <u>Y</u> respt., as in Fig 19. We may consider the lengths CX and CY as the coordinates of <u>M</u>. This system wae invented by Lemoine.

(l) <u>Parallel</u>. A system of point coordinates, similar to the system of parallel line coordinates already considered, may be established as follows. Fix two parallel lines <u>p</u> and <u>q</u> and a line AB perpendicular to them and meeting them in the points <u>A</u> and <u>B</u>; see Fig. 20. From any point <u>P</u> draw the lines PB and PA and let these lines meet the lines <u>p</u> and <u>q</u> in the points <u>U</u> and <u>V</u>. The reciprocal values of AU and BV may be considered as the (p, q) coordinates of <u>P</u>. These values are called the <u>parallel</u> coordinates of <u>P</u>. If AB is taken as the <u>X</u>-axis and is two units in length and if as the <u>Y</u>-axis is taken the perpendicular bisector of AB, then these parallel coordinates (p, q) will be connected with the rectangular coordinates (x, y) by the equations

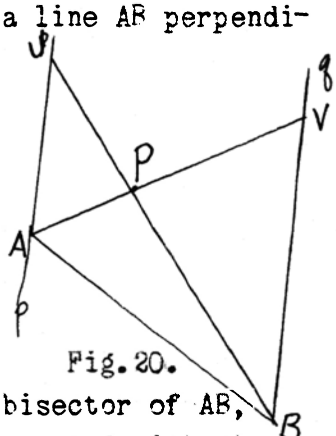


Fig. 20.

$$p = \frac{1 \quad x}{2y}$$

$$q = \frac{1 \quad x}{2y}$$

[1] Franklin Am. Jour. of Math. XII (1890) 161 90.
Poulain Jahr. Fort. der M. XXI (1889) 681.
Lemoine " " " " XX (1888): 683.
D'Ocagne " " " " XIX (1887): 692.
" Nouvelles Annalles (3) VI (1898) 493 502.